# Three wave interaction in pair plasmas

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It is known that an electron-ion plasma with a density gradient perpendicular to the ambient magnetic field vector supports drift waves driven by the density gradient. The vector-product-type nonlinearity leads to a three-wave interaction which allows for both a direct and an inverse cascade, i.e., the transport of energy towards both shorter and longer wavelengths. In the case of pair plasmas (electron-positron, or pair ion plasmas [1, 2]), instead of the drift mode one obtains convective cells [3]. In the present work it is shown that nonlinear three-wave interaction [4], described by vector-product type nonlinearities, in pair plasmas implies much more restrictive conditions for a double energy transfer, as compared to electron-ion plasmas.

We assume a plasma with two components of equal mass and opposite charge (electron-positron or pair-ion), and we use the continuity and momentum equations in the form

$$\frac{\partial}{\partial t}\left(\frac{n_1}{n_0}\right)+\nabla_\perp\cdot\vec{v}_{\perp 1}+\frac{\partial v_{z1}}{\partial z}+\vec{v}_{\perp 1}\cdot\nabla_\perp n_0+\frac{n_1}{n_0}\nabla\cdot\vec{v}_1+\vec{v}_1\frac{\nabla n_1}{n_0}=0, \tag{1}$$

$$\left(\frac{\partial}{\partial t}+\vec{v}_1\cdot\nabla\right)\vec{v}_1=\frac{q}{m}\left(-\nabla\phi_1+\vec{v}_1\times\vec{B}_0\right). \tag{2}$$

The indices 0 and 1 here denote equilibrium and perturbed quantities, respectively. We study low frequency perturbations $\sim\exp(-i\omega t+i\vec{k}_\perp\vec{r}+ik_z z)$, $|\partial/\partial t|\ll|\Omega|=|q|B_0/m$, propagating nearly perpendicularly with respect to the magnetic field vector $\vec{B}_0=B_0\vec{e}_z$, i.e., $|k_\perp|\gg|k_z|$, and we allow for the presence of the equilibrium perpendicular density gradient $\nabla_\perp n_0$. It turns out that the density gradient effects play no role in the present problem because the corresponding terms for the two species cancel out. From Eq. (2) we obtain approximately

$$\vec{v}_{\perp 1}=\frac{1}{B_0}\vec{e}_z\times\nabla_\perp\phi_1-\frac{1}{B_0\Omega}\left(\frac{\partial}{\partial t}+\frac{1}{B_0}\vec{e}_z\times\nabla_\perp\phi_1\cdot\nabla_\perp\right)\nabla_\perp\phi_1, \tag{3}$$

$$\left(\frac{\partial}{\partial t}+\frac{1}{B_0}\vec{e}_z\times\nabla_\perp\phi_1\cdot\nabla_\perp\right)v_z=-\frac{q}{m}\frac{\partial\phi_1}{\partial z}. \tag{4}$$

Using these two in Eq. (1), after some algebra one obtains

$$\left(\frac{\partial}{\partial t}+\frac{1}{B_0}\vec{e}_z\times\nabla_\perp\phi_1\cdot\nabla_\perp\right)\left[\left(\frac{\partial}{\partial t}+\frac{1}{B_0}\vec{e}_z\times\nabla_\perp\phi_1\cdot\nabla_\perp\right)\nabla_\perp^2\phi_1\right]=-\Omega^2\frac{\partial^2\phi_1}{\partial z^2}. \tag{5}$$

$$\mathscr{L}\equiv\frac{\partial}{\partial t}+\frac{1}{B_0}\vec{e}_z\times\nabla_\perp\phi_1\cdot\nabla_\perp.$$

Linearizing Eq. (5) one then obtains the dispersion equation for the electrostatic convective cells propagating almost perpendicularly to the magnetic field vector:

$$\omega^2 = \Omega^2 \frac{k_z^2}{k_\perp^2}. \tag{6}$$

Note that using the Poisson equation instead of the quasi-neutrality will only modify the constant on the right-hand side of Eqs. (5, 6), $\Omega^2 \to \Omega^2/[1+\Omega^2/(2\omega_p^2)]$, $\omega_p^2 = q^2 n_0/(\varepsilon_0 m)$. This modification is not essential for our work and will be neglected implying that $\Omega^2 \ll 2\omega_p^2$.

In order to study the three-wave interaction, we use the nonlinear Eq. (5) assuming the perturbed potential (after omitting the previously used index 1) in the form

$$\phi(t) = \sum_{j=1}^{j=3} \left[ \Phi_j(t) \exp\left(-i\omega_j t + i\vec{k}_j \vec{r}\right) + \Phi_j^*(t) \exp\left(i\omega_j t - i\vec{k}_j \vec{r}\right) \right]. \tag{7}$$

Here, $*$ denotes the complex-conjugate quantity, and we shall use also $\partial/\partial t|_j \to -i\omega_j + \delta/\delta t$, $|i\omega_j| \gg |\delta/\delta t|$, where $\delta/\delta t$ is the time derivative on a slow time-scale, as a consequence of the nonlinear three-wave interaction. In this case one obtains the following expression for time variation of the $j$th-amplitude:

$$\frac{\delta \Phi_j(t)}{\delta t} = \frac{i}{2\omega_j k_j^2} \left\{ \frac{1}{B_0} \left( \vec{e}_z \times \nabla_\perp \frac{\partial \phi}{\partial t} \cdot \nabla_\perp \right) \nabla_\perp^2 \phi + \frac{2}{B_0} \left( \vec{e}_z \times \nabla_\perp \phi \cdot \nabla_\perp \right) \frac{\partial \nabla_\perp^2 \phi}{\partial t} \right.$$

$$\left. + \frac{1}{B_0^2} \left( \vec{e}_z \times \nabla_\perp \phi \cdot \nabla_\perp \right) \left[ \left( \vec{e}_z \times \nabla_\perp \phi \cdot \nabla_\perp \right) \nabla_\perp^2 \phi \right] \right\}. \tag{8}$$

On the right hand side in Eq. (8), $\phi$ should be taken as the summation (7). In view of the approximative calculation of the time-varying mode amplitude, on the right-hand side the remaining time and space derivatives $\partial/\partial t, \nabla_\perp$ imply $\pm i\omega_l, \pm \vec{k}_l$ $(l = 1,2,3)$, where $\pm$ appears due to complex-conjugate expressions, and out of all terms introduced by the summation (7), one should keep only the resonant ones, corresponding to the $-i\omega_j t + i\vec{k}_j \cdot \vec{r}$ on the left-hand side. Without any loss of generality we further assume the following resonant conditions:

$$\omega_1 = \omega_2 + \omega_3, \quad \vec{k}_1 = \vec{k}_2 + \vec{k}_3. \tag{9}$$

The remaining task of calculating the nonlinear terms in Eq. (8) is presented below. For the mode $\omega_1, \vec{k}_1$, from the first term on the right-hand side in Eq. (8) we have

$$\Gamma_{1,1} \equiv \frac{1}{B_0} \left( \vec{e}_z \times \nabla_\perp \frac{\partial \phi}{\partial t} \cdot \nabla_\perp \right) \nabla_\perp^2 \phi \Rightarrow \frac{i}{B_0} \vec{e}_z \cdot (\vec{k}_3 \times \vec{k}_2)(\omega_2 k_3^2 - \omega_3 k_2^2) \Phi_2 \Phi_3.$$

Similarly, the second nonlinear term yields:

$$\Gamma_{1,2} \equiv \frac{2}{B_0} \left( \vec{e}_z \times \nabla_\perp \phi \cdot \nabla_\perp \right) \frac{\partial \nabla_\perp^2 \phi}{\partial t} \Rightarrow \frac{2i}{B_0} \vec{e}_z \cdot (\vec{k}_3 \times \vec{k}_2)(\omega_3 k_3^2 - \omega_2 k_2^2) \Phi_2 \Phi_3.$$

Consequently, Eq. (8) for the mode $\omega_1, \vec{k}_1$ becomes

$$\frac{\delta\Phi_1(t)}{\delta t} - \Upsilon_{13} = -\frac{\vec{e}_z\cdot(\vec{k}_3\times\vec{k}_2)}{2B_0\omega_1 k_1^2}\left[\omega_2 k_3^2 - \omega_3 k_2^2 + 2(\omega_3 k_3^2 - \omega_2 k_2^2)\right]\Phi_2\Phi_3. \tag{10}$$

The meaning of the term $\Upsilon_{13}$ is obvious from Eq. (8), it is the third nonlinear term there.

In the same manner from Eq. (8) one obtains for the modes 2 and 3, respectively:

$$\frac{\delta\Phi_2(t)}{\delta t} - \Upsilon_{23} = -\frac{\vec{e}_z\cdot(\vec{k}_1\times\vec{k}_3)}{2B_0\omega_2 k_2^2}\left[\omega_1 k_3^2 + \omega_3 k_1^2 - 2(\omega_3 k_3^2 + \omega_1 k_1^2)\right]\Phi_1\Phi_3^*. \tag{11}$$

$$\frac{\delta\Phi_3(t)}{\delta t} - \Upsilon_{33} = -\frac{\vec{e}_z\cdot(\vec{k}_1\times\vec{k}_2)}{2B_0\omega_3 k_3^2}\left[\omega_1 k_2^2 + \omega_2 k_1^2 - 2(\omega_2 k_2^2 + \omega_1 k_1^2)\right]\Phi_1\Phi_2^*. \tag{12}$$

Equations (10)-(12) describe time evolution of the three modes due to their nonlinear interaction. For an arbitrary $j$th mode on the left-hand sides in Eqs. (10)-(12), the first and second nonlinear terms on the right-hand sides must include terms $k$ and $l$, where $k \neq l$ (including the combination with the complex-conjugate counterparts too).

On the other hand, the third nonlinear term $\Upsilon_{j3}$ includes products of all three amplitudes $\Phi_j, \Phi_k, \Phi_l$, and consequently, it can not contain resonant exponential terms that would follow from the interaction between different modes. In other words, for the $j$th mode on the left-hand sides in Eqs. (10)-(12), the third nonlinear term $\Upsilon_{j3}$ can only contain the self-interacting terms of the type $c_j \cdot \Phi_j$, where in the same time the interaction coefficient can only include terms of the form $\Phi_k\Phi_k^*$, $\Phi_l\Phi_l^*$, where $k \neq j$, and $l \neq j$. Hence, although it describes the variation of the mode amplitude due to 3-wave interaction, it does not directly contribute to a possible energy transfer towards larger and shorter wave-lengths. It is a cubic nonlinearity yielding only a frequency shifts due to the modulational interaction. As such it in principle introduces a mismatch in the perfect resonant condition (9) for the frequencies, yet these effects will not be discussed here.

The double energy transfer can follow only from the first and second nonlinear terms in Eq. (8), therefore only the contribution of those terms will be checked below. For that purpose, using the resonant conditions (9) we rewrite Eqs. (10)-(12) in a more symmetric form

$$\frac{\delta\Phi_1(t)}{\delta t} - \Upsilon_{13} = \frac{\vec{e}_z\cdot(\vec{k}_2\times\vec{k}_3)}{2B_0\omega_1 k_1^2}\left[\omega_2 k_3^2 - \omega_3 k_2^2 + 2(\omega_3 k_3^2 - \omega_2 k_2^2)\right]\Phi_2\Phi_3. \tag{13}$$

$$\frac{\delta\Phi_2(t)}{\delta t} - \Upsilon_{23} = \frac{\vec{e}_z\cdot(\vec{k}_2\times\vec{k}_3)}{2B_0\omega_2 k_2^2}\left[-\omega_1 k_3^2 - \omega_3 k_1^2 + 2(\omega_3 k_3^2 + \omega_1 k_1^2)\right]\Phi_1\Phi_3^*. \tag{14}$$

$$\frac{\delta\Phi_3(t)}{\delta t} - \Upsilon_{33} = \frac{\vec{e}_z\cdot(\vec{k}_2\times\vec{k}_3)}{2B_0\omega_3 k_3^2}\left[\omega_1 k_2^2 + \omega_2 k_1^2 - 2(\omega_2 k_2^2 + \omega_1 k_1^2)\right]\Phi_1\Phi_2^*. \tag{15}$$

The double energy transfer implies an energy transfer towards both shorter (the direct one) and longer scales (the inverse transfer). Taking as an example

$$k_2^2 < k_3^2 < k_1^2, \tag{16}$$

for positive frequencies and after using (9) to eliminate $\omega_3$ (after disregarding the obvious positive and common terms) the signs of the coefficients of interaction are determined by:

$$\beta_1 = k_3^2(2\omega_1 - \omega_2) - k_2^2(\omega_1 + \omega_2), \tag{17}$$

$$\beta_2 = k_3^2(\omega_1 - 2\omega_2) + k_1^2(\omega_1 + \omega_2), \tag{18}$$

$$\beta_3 = k_2^2(\omega_1 - 2\omega_2) + k_1^2(\omega_2 - 2\omega_1). \tag{19}$$

In order to have the double energy transfer with the intermediate mode yielding the energy, $\beta_3 < 0$, $\beta_{1,2} > 0$, from Eqs. (17)-(19) we obtain the following additional conditions for the frequencies and wave-numbers:

$$\omega_1 > 2\omega_2, \quad k_1^2 > k_2^2 \frac{\omega_1 - 2\omega_2}{2\omega_1 - \omega_2}, \quad k_3^2 > k_2^2 \frac{\omega_1 + \omega_2}{2\omega_1 - \omega_2}. \tag{20}$$

Hence, the same double energy transfer in the pair-plasma imposes three more conditions. Yet, it is still possible and as an example we take the following set of numbers: $\omega_1 = 3$, $\omega_2 = 1$, $\omega_3 = 2$ and $k_1^2 = 3$, $k_2^2 = 1$, $k_3^2 = 2$. These indeed satisfy all the conditions (9), (16), (20) and allow for the double energy transfer because: $\beta_1 = 6$, $\beta_2 = 14$, $\beta_3 = -14$. From the derivations, and in particular from Eq. (8), it is clear that the $k_j$ here imply the perpendicular components of the wave numbers. On the other hand, because both $\Omega$ and the ratio $k_z/k_\perp$ remain free (except for the assumption that the latter is much less than unity), the dispersion relation (6) will easily be satisfied too. We stress again that pair properties imply that the results presented here are valid for both homogeneous and inhomogeneous environments.

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